3401. (a) Since it is coming from the West at 5 miles per hour, the bear's position can be expressed as $\mathbf{r}_b = (5t + c)\mathbf{i}$. We want the bear to arrive at $t = 0.1$ hours: $5 \times 0.1 + c = 0$ gives $c = -1/2$. The hiker's position is $\mathbf{r}_h = 3t\mathbf{j}$.

(b) The squared distance is $S = (5t - 1/2)^2 + (3t)^2$. This simplifies to $S = \frac{1}{4}(136t^2 - 20t + 1)$.

(c) Looking for the time at which S is stationary:

$$\begin{aligned} \tfrac{dS}{dt} &= \tfrac{1}{4}(272t - 20) = 0 \\ \Longrightarrow t &= \tfrac{5}{68}. \end{aligned}$$

At this time, the squared distance S is $\frac{9}{136}$. So, in miles, the minimum distance between bear and hiker is $\sqrt{S} = 0.257... > 1/4$, as required.

3402. The behaviour for odd and even cases is different. The ranges are

(a) $[0, b^2]$,
(b) $[-a^3, b^3]$,
(c) $[0, b^4]$.

3403. The next move for × is e.g. the top-right corner, which forces ○ to respond with bottom-left. Then × plays bottom-right, which forces ○ to respond with top-left. Then × plays middle-right to win.

3404. Let $u = 1 + \ln x$. Then $du = \frac{1}{x}dx$, which appears directly in the integral. Also, we have $\ln x = u - 1$. Enacting the substitution,

$$\begin{aligned} &\int \frac{\ln x}{x(1 + \ln x)^2}\,dx \\ =& \int \frac{u-1}{u^2}\,du \\ \equiv& \int u^{-1} - u^{-2}\,du \\ \equiv&\ \ln|u| + u^{-1} + c \\ =&\ \ln|1 + \ln x| + \frac{1}{1 + \ln x} + c. \end{aligned}$$

3405. (a) The boundary equation is $\sin^2 x + \sin^2 y = 1$. Doubled, this is $2\sin^2 x + 2\sin^2 y = 2$. Using the $\cos 2\theta \equiv 1 - 2\sin^2\theta$ double-angle identity,

$$\begin{aligned} 1 - \cos 2x + 1 - \cos 2y &= 2 \\ \Longrightarrow \cos 2x + \cos 2y &= 0. \end{aligned}$$

(b) We can rewrite as $\cos 2x = -\cos 2y$. Adding π to the input of the cos (or sin) function negates the output. Hence, $\cos 2x = \cos(2y + \pi)$.

(c) The primary values for which the boundary equation is satisfied are

$$\begin{aligned} 2x &= 2y + \pi, \\ 2x &= 2\pi - (2y + \pi). \end{aligned}$$

These simplify to $x - y = \pi/2$ and $x + y = \pi/2$. But $\cos 2x$ and $\cos 2y$ both have period π, so the addition of $m\pi, n\pi$ to either coordinate produces another valid solution. This gives the required result.

(d) The lines form an infinite mosaic of squares. Every other square is red, in a chequerboard pattern. Hence, the probability is $\frac{1}{2}$.

3406. We take out a common factor of x^2, then factorise a quadratic in x^4:

$$\begin{aligned} x^{10} - 6x^6 + 9x^2 &= 0 \\ \Longrightarrow x^2(x^8 - 6x^4 + 9) &= 0 \\ \Longrightarrow x^2(x^4 - 3)^2 & \\ \Longrightarrow x &= 0, \pm\sqrt[4]{3}. \end{aligned}$$

So, the equation has three roots.

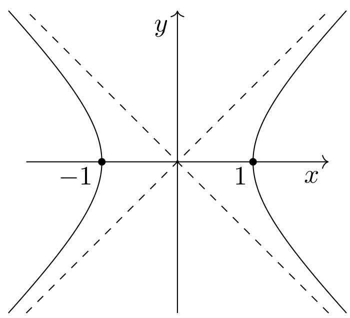
3407. (a) Differentiating with respect to y,

$$\begin{aligned} x^2 - y^2 &= 1 \\ \Longrightarrow 2x\tfrac{dx}{dy} - 2y &= 0. \end{aligned}$$

Setting $dx/dy = 0$ for vertical tangents, we get $y = 0$. Substituting back in gives $(\pm 1, 0)$.

(b) Factorising, $(x + y)(x - y) = 1$. If either of the factors on the LHS approaches zero, then the other factor grows without bound. Hence, $x + y = 0$ and $x - y = 0$ are the asymptotes.

(c) The only axis intercepts are the points found in (a). Together with the asymptotes from (b), this gives



3408. If either one of a or x is negative or zero, then at least one side of the identity is undefined for even n. For this reason, logarithms are usually defined over a positive base. The conditions are $a > 0, x > 0$.

Nota Bene

The identity could technically be said to hold for $a < 0, x < 0$ and n odd. But this would be a rather unnatural interpretation.

3409. The curves are reflections of one another in the line $y = x$. So, by symmetry, the shortest path between them must be symmetrical in $y = x$.

The curves do not intersect, so neither can they intersect $y = x$. Hence, the shortest path cannot be $y = x$ itself.

The shortest path must therefore be normal to $y = x$, i.e. of the form $x + y = k$. □

3410. We expand binomially. All but the first two terms of each expansion have factors of h^2 and above, and won't contribute to the limit:

$$\begin{aligned}
&\lim_{h\to 0} \frac{(x+h)^a - x^a}{(x+h)^b - x^b} \\
\equiv\ &\lim_{h\to 0} \frac{x^a + ax^{a-1}h + ... - x^a}{x^b + bx^{b-1}h + ... - x^b} \\
\equiv\ &\lim_{h\to 0} \frac{ax^{a-1}h + ...}{bx^{b-1}h + ...} \\
\equiv\ &\lim_{h\to 0} \frac{ax^{a-1} + ...}{bx^{b-1} + ...} \\
\equiv\ &\tfrac{a}{b}x^{a-b}.
\end{aligned}$$

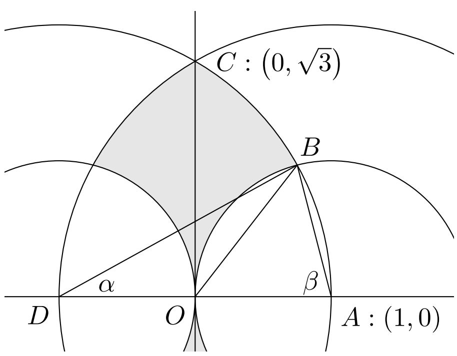
3411. (a) The derivatives are

$$\begin{aligned}
x &= e^{-t} + \sin t \\
\frac{dx}{dt} &= -e^{-t} + \cos t \\
\frac{d^2x}{dt^2} &= e^{-t} - \sin t \\
\frac{d^3x}{dt^3} &= -e^{-t} - \cos t.
\end{aligned}$$

Adding these four equations, the exponentials and sinusoids all cancel to give zero. Hence, $x = e^{-t} + \sin t$ satisfies the DE.

(b) The exponential term tends to zero. So, the position tends towards a sinusoidal oscillation: $x \to \sin t$.

3412. The boundary equations are four circles, centred on $(\pm 1, 0)$, with radii 1 and 2. Calculating the y intercept, the scenario is:



$\triangle ABD$ has side lengths $(1, 2, 2)$, so $\alpha = \arccos\frac{7}{8}$, and $\beta = \arccos\frac{1}{4}$.

Consider the positive quadrant. The region OAC in common to the two large circles is a half-segment, which subtends $\frac{\pi}{3}$ radians at D. It has area $\frac{2\pi}{3} - \frac{\sqrt{3}}{2} = 1.22836...$

Next, region OAB. This is segment AB, radius 2, and sector OAB, radius 1. Segment AB subtends α at D, with area $2(\alpha - \sin\alpha) = 0.04247...$; sector AOB subtends β at A, with area $\frac{1}{2}\beta = 0.65905...$.

So, the total area in all four quadrants is

$$A = 4(1.22836 - 0.65905 - 0.04247) = 2.11 \text{ (3sf)}.$$

3413. (a) Using $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$, $\mathbb{P}(A \cup B) = 1 - 1/3 = 2/3$. Subtracting this from 1 gives $\mathbb{P}(A' \cap B') = 1/3$.

(b) We can express $\mathbb{P}(\text{exactly one of } A \text{ or } B)$ as $\mathbb{P}(A \cup B) - \mathbb{P}(A \cap B) = 1/3$.

3414. Setting $x^2 + 4x + 7 = x$ gives $x^2 + 3x + 7 = 0$. This has $\Delta = -19 < 0$, so the iteration has no fixed points. Since $x^2 + 3x + 7$ is a positive quadratic, its value is always greater than x. Hence, the output is always greater than the input of the iteration, and it is therefore always increasing.

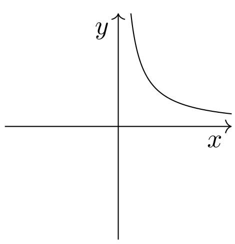
3415. We can express the RHS as $\cos 3x = \cos(2x + x)$, and expand with a compound-angle formula and a double-angle formula:

$$\begin{aligned}
&\cos x - \cos 3x \\
\equiv\ &\cos x - (\cos 2x \cos x - \sin 2x \sin x) \\
\equiv\ &\cos x - \cos x(1 - 2\sin^2 x) + \sin 2x \sin x \\
\equiv\ &2\cos x \sin^2 x + \sin 2x \sin x \\
\equiv\ &(2 \sin x \cos x)\sin x + \sin 2x \sin x \\
\equiv\ &2 \sin x \sin 2x, \text{ as required.}
\end{aligned}$$

3416. Rearranging the algebra,

$$\begin{aligned}
&\log_x a + \log_y a = 0 \\
\implies &-\log_y a = \log_x a \\
\implies &\log_a y = -\log_a x \\
\implies &\log_a y = \log_a \tfrac{1}{x} \\
\implies &y = \tfrac{1}{x}.
\end{aligned}$$

So, the graph $\log_x a + \log_y a = 0$ is independent of a: it is the standard reciprocal. However, the original logarithms are only defined for $x, y > 0$:



3417. Rearranging the equation of the curve to $x = y^2$, we set up a definite integral with respect to y:

$$\begin{aligned} &\int_0^k y^2\, dy = 21 \\ \Longrightarrow\ &\left[\tfrac{1}{3}y^3\right]_0^k = 21 \\ \Longrightarrow\ &\tfrac{1}{3}k^3 = 72 \\ \Longrightarrow\ &k = 6. \end{aligned}$$

3418. The condition given restricts the possibility space to four outcomes: HTT, THT, TTH and TTT. Of these, one is all tails. So, the probability is 1/4.

3419. (a) False: $(2, 4)$ is a counterexample.
(b) False: $(2, 4)$ is a counterexample.
(c) True: $\text{hcf}(x, y) = 1$ but $\min(x, y) > 1$.

3420. The line $y = x + k$ is parallel to the line $y = x$. So, reflection in $y = x + k$ is equivalent to reflection in $y = x$, followed by translation by vector $-k\mathbf{i} + k\mathbf{j}$.

Reflecting $y = \text{f}(x)$ in $y = x$, we switch x and y to give $x = \text{f}(y)$. Translating, we replace x by $x + k$ and y by $y - k$. The new graph is $x + k = \text{f}(y - k)$, as required.

— Nota Bene —

To visualise the above, consider the effect on $(0, 0)$ of reflection in $y = x + 4$.

3421. Consider reflecting $y = \text{f}(x)$ in the line $x = a$. The point $\big(a, \text{f}(a)\big)$ in fixed by such a reflection, and the gradient is negated, becoming $-\text{f}'(a)$. Since $y = \text{f}(x)$ is symmetrical, this must be equal to the original gradient at a:

$$\begin{aligned} &\text{f}'(a) = -\text{f}'(a) \\ \Longrightarrow\ &\text{f}'(a) = 0. \end{aligned}$$

So, if $y = \text{f}(x)$ is a polynomial graph with a line of symmetry at $x = a$, then $\text{f}'(a) = 0$. □

3422. Let $u^2 = x - 1$. Then $2u\frac{du}{dx} = 1$, so $dx = 2u\, du$. Taking the positive square root, this gives

$$\begin{aligned} &\int_5^{17} \frac{1}{(x-1)(1-\sqrt{x-1})}\, dx \\ =\ &\int_2^4 \frac{1}{u^2(1-u)} 2u\, du \\ =\ &\int_2^4 \frac{2}{u(1-u)}\, du \end{aligned}$$

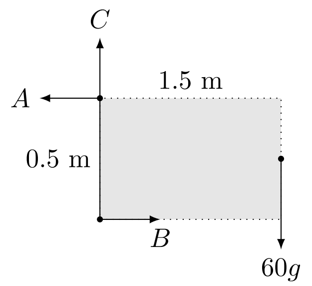
Writing in partial fractions,

$$\frac{2}{u(1-u)} \equiv \frac{2}{u} + \frac{2}{1-u}.$$

So, the integral is

$$\begin{aligned} &\int_2^4 \frac{2}{u} + \frac{2}{1-u}\, du \\ &= 2\Big[\ln|u| - \ln|1-u|\Big]_2^4 \\ &= 2(\ln 4 - \ln 3) - 2(\ln 2 - \ln 1) \\ &= \ln 16 - \ln 9 - \ln 4 \\ &= \ln \tfrac{4}{9}, \text{ as required.} \end{aligned}$$

3423. (a) Modelling the gymnast as a rectangle (shaded below), we express the contact forces on his hands in horizontal (each hand separate) and vertical (both hands combined) components. The force diagram is

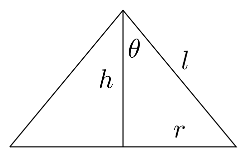


Vertically, $C = 60g$. Horizontally, $A - B = 0$. Taking moments around the midpoint of his hands, $0.25A + 0.25B = 1.5 \cdot 60g$. Solving, $A = B = 180g$. Horizontally, the top hand is in tension; the bottom hand is in thrust.

(b) Since the vertical (frictional) forces on each hand have the same line of action, there is no distinction between them in the equations above. Combined, they have magnitude $60g$ N, but that's all we know.

3424. Since $\text{g}(x)$ is decreasing everywhere, the graph $y = \text{g}(x)$ can have no SPs. Every polynomial graph of even degree has at least one SP, so g is of odd degree. And every polynomial of odd degree has range $\mathbb{R}$. □.

3425. The surface area is $A = \pi r(l + r)$, where l is the slant height. In cross-section, the cone is



Using trigonometry, $r = h\tan\theta$ and $l = \text{h}\sec\theta$. Substituting these into the area formula,

$$\begin{aligned} A &= \pi r(l + r) \\ &= \pi h^2 \tan\theta(\tan\theta + \sec\theta), \text{ as required.} \end{aligned}$$

3426. Simplifying with a log rule,

$$\begin{aligned}&\ln {}^{2k}\mathrm{C}_k - \ln {}^{2k}\mathrm{C}_{k+1}\\ \equiv\ &\ln \frac{{}^{2k}\mathrm{C}_k}{{}^{2k}\mathrm{C}_{k+1}}.\end{aligned}$$

Expressing ${}^n\mathrm{C}_r$ in factorials, this is

$$\begin{aligned}&\ln \frac{\frac{(2k)!}{k!k!}}{\frac{(2k!)}{(k+1)!(k-1)!}}\\ \equiv\ &\ln \frac{(k+1)!(k-1)!}{k!k!}\\ \equiv\ &\ln \tfrac{k+1}{k},\text{ as required.}\end{aligned}$$

3427. Writing longhand, the function is

$$\mathrm{f}(x) = \frac{3}{x(x-1)} + \frac{3}{(x-1)(x-2)} + \frac{3}{(x-2)(x-3)}.$$

We simplify, and then complete the square:

$$\mathrm{f}(x) = \frac{9}{x^2-3x} = \frac{1}{\frac{1}{9}\left(x-\frac{3}{2}\right)^2 - \frac{1}{4}}.$$

The denominator has range $[-\frac{1}{4}, \infty)$. So, f has range $(-\infty, -4] \cup (0, \infty)$.

3428. If the decimal expansion of x terminates after n digits, then x can be expressed as $x = \frac{a}{10^n}$, where $a, n \in \mathbb{N}$. This fraction can then be expressed in its lowest terms, by dividing top and bottom by factors of 2 and/or 5. Once all common factors have been cancelled, we have expressed x in the required form. QED.

3429. (a) Differentiating with respect to y,

$$2x\frac{dx}{dy}(y+1)^2 + 2x^2(y+1) = 1.$$

Setting $\frac{dx}{dy} = 0$ for vertical tangents, we get $2x^2(y+1) = 1$, so $x^2 = \frac{1}{2(y+1)}$. Substituting into the original relation,

$$\begin{aligned}&\frac{1}{2(y+1)}(y+1)^2 = y\\ \implies &\tfrac{1}{2}(y+1) = y\\ \implies &y = 1.\end{aligned}$$

So, the tangent is parallel to $\mathbf{j}$ at $(\pm 1/2, 1)$.

(b) Rearranging to make x^2 the subject,

$$x^2 = \frac{y}{(y+1)^2}.$$

Since the degree of the denominator is larger than that of the numerator, the quotient tends to zero as $y \to \infty$. Hence $x^2 \to 0$, and both branches tend asymptotically to the y axis, as required.

3430. Differentiating, the velocity is

$$\mathbf{v} = \begin{pmatrix}\cos t\\ -2\sin 2t\\ 2\cos 2t\end{pmatrix} \text{ ms}^{-1}.$$

The speed is 2 ms^{-1}, so

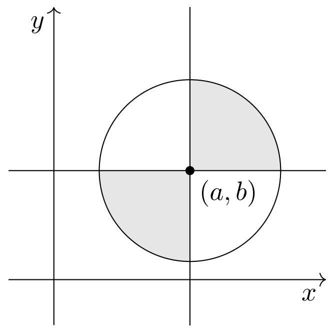
$$\begin{aligned}&\cos^2 t + 4\sin^2 2t + 4\cos^2 t = 4\\ \implies &\cos^2 t + 4 = 4\\ \implies &\cos t = 0.\end{aligned}$$

The first value $t > 0$ is $t = \frac{\pi}{2}$ s.

3431. We know that $\mathrm{f}(x)$ is convex on $x \in (\infty, a) \cup (b, \infty)$. Putting this algebraically, $\mathrm{f}''(x) > 0$ has solution $x \in (\infty, a) \cup (b, \infty)$.

Hence, the boundary equation $\mathrm{f}''(x) = 0$ must have distinct roots at $x = a$ and $x = b$. So, f'' must have degree at least two. Integrating twice, f must have degree at least 4. □

3432. The boundary equations are $(x-a)(y-b) = 0$, which is the pair of perpendicular lines $x = a$ and $y = b$, and $(x-a)^2 + (y-b)^2 = 1$, which is a unit circle centred on (a, b). So, the region is



— Nota Bene —

To see why the shading for $(x-a)(y-b) \geq 0$ takes a chequerboard pattern, note that, for a product to be positive, either both factors must be positive or both negative.

3433. Using a double-angle formula and then multiplying both sides by $\tan a(1 - \tan^2 a)$,

$$\begin{aligned}&\tan 2a - \cot a = 0\\ \implies &\frac{2\tan a}{1-\tan^2 a} = \cot a\\ \implies &2\tan^2 a = 1 - \tan^2 a\\ \implies &\tan a = \pm\tfrac{1}{\sqrt{3}}\\ \therefore\quad &a = \tfrac{\pi}{6}, \tfrac{5\pi}{6}, \tfrac{7\pi}{6}, \tfrac{11\pi}{6}.\end{aligned}$$

3434. Equating ratios in the GP,

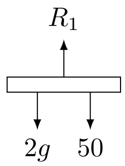
$$\begin{aligned}&\frac{a}{100} = \frac{b}{a}\\ \implies &100b = a^2.\end{aligned}$$

Equating differences in the AP,

$$\begin{aligned} a(b-1) - 100 &= a - a(b-1) \\ \implies 100 &= a(2b-3). \end{aligned}$$

Solving simultaneously, $a = 20$ and $b = 4$.

3435. (a) The force diagram for the top box is



So, $R_1 = 50 + 2g = 69.6$ N.

(b) Consider the entire stack as one object. The binding forces cancel, and the reaction force from the floor is $2ng$. The 50 N binding also acts on the bottom box, so $R_n = 50 + 2ng$ N.

(c) As seen from parts (a) and (b), the bottom box experiences the greatest contact force, on its lower face. So, we solve $50 + 2ng = 500$, which gives $n = 22.9$. Hence, the greatest number of boxes which can be safely stacked is $n = 22$.

3436. Each equation has $x = 0$ as a root. Taking this out, we are left with

① $x^4 + x^2 - 1 = 0$. This is a quadratic in x^2 with positive discriminant. It has two real roots for x^2. One of these is positive, which then gives two real roots for x:

$$\begin{aligned} x^2 &= \tfrac{-1+\sqrt{5}}{2} \\ \implies x &= \pm\sqrt{\tfrac{-1+\sqrt{5}}{2}}. \end{aligned}$$

② $x^4 + x^2 + 1 = 0$. This has $\Delta < 0$, so no real roots for x^2 or therefore for x.

Hence, statement ① is false: the roots given above are counterexamples.

3437. To get the second graph from the first, we replace y by $y - p$. This is a translation by vector $p\mathbf{j}$.

3438. The product rule is

$$\frac{d}{dx}(uv) = \frac{du}{dx}v + u\frac{dv}{dx}.$$

Firstly, we rearrange to

$$u\frac{dv}{dx} = \frac{d}{dx}(uv) - \frac{du}{dx}v.$$

Then, we integrate both sides with respect to x:

$$\int u\frac{dv}{dx}\,dx = uv - \int \frac{du}{dx}v\,dx.$$

3439. The equation of motion for the system along the string is $2mg - mg = 3ma$. Solving this, each load accelerates vertically at $a = \frac{1}{3}g$.

The relative acceleration is therefore $\frac{2}{3}g$. And the initial relative speed is 0. Using $s = ut + \frac{1}{2}at^2$, the relative position is given by $d = \frac{1}{3}t^2$.

3440. Raising base and input to the same power of $\frac{1}{2}$, the second term is $\log_2(\sqrt{2} \times 4^x)$. Writing everything over base 2,

$$\begin{aligned} & \log_2\left(2^2 \times 2^{3x}\right) - \log_2\left(2^{\frac{1}{2}} \times 2^{2x}\right) = 1 \\ \implies & \log_2\left(2^{2+3x}\right) - \log_2\left(2^{\frac{1}{2}+2x}\right) = 1 \\ \implies & 2 + 3x - \left(\tfrac{1}{2} + 2x\right) = 1 \\ \implies & x = -\tfrac{1}{2}. \end{aligned}$$

3441. The implication is ① $\implies$ ②. If a polynomial function f is increasing on $\mathbb{R}$, then it must have odd degree, range $\mathbb{R}$, and it must be one-to-one. So, it is invertible.

But the function $\mathrm{f}(x) = -x^3$ is invertible, but not increasing: it is invertible because it is *decreasing* (almost) everywhere.

3442. Differentiating implicitly,

$$\begin{aligned} 2x + 2y\frac{dy}{dx} &= 0 \\ \implies \frac{dy}{dx} &= -\frac{x}{y}. \end{aligned}$$

Differentiating again by the quotient rule,

$$\begin{aligned} \frac{d^2y}{dx^2} &= \frac{-y + x\frac{dy}{dx}}{y^2} \\ &= \frac{-y - x \cdot -\frac{x}{y}}{y^2} \\ &\equiv \frac{x^2 - y^2}{y^3}, \text{ as required.} \end{aligned}$$

3443. Assume, for a contradiction, that $\mathrm{f}(x) - \mathrm{g}(x) = 0$ is a cubic equation. The graphs cross each other twice, so this equation has exactly two single roots. But taking out two single factors would leave a third, corresponding to a third crossing point. This is a contradiction.

So, $\mathrm{f}(x) - \mathrm{g}(x) = 0$ is not a cubic equation. Since f and g are cubics, this can only happen if they have the same leading coefficient. □

3444. (a) This is an output transformation: translation by $3\mathbf{j}$, followed by a stretch factor 2 in the y direction.

(b) This is an input transformation: translation by $-3\mathbf{i}$, followed by a stretch factor $\frac{1}{2}$ in the x direction.

(c) Since gh is an invertible function, applying it to both sides of the equation $y = \mathrm{f}(x)$ leaves the graph unchanged.

3445. Assuming that the quadratic divides in exactly, we look to express the quartic as

$$(x^2 + x + 8)(ax^2 + bx + c).$$

Equating coefficients,

$$\begin{aligned} x^4 &: a = 1 \\ x^3 &: a + b = -1 \implies b = -2 \\ x^2 &: 8a + b + c = 16 \implies c = 10. \end{aligned}$$

The remaining terms match, so x^2+x+8 is indeed a factor. The quotient is $x^2 - 2x + 10$.

— Alternative Method —

Using polynomial long division,

$$\begin{array}{r|l} & x^2 - 2x + 10 \\ \hline x^2 + x + 8 & x^4 - x^3 + 16x^2 - 6x + 80 \\ & -x^4 - x^3 - 8x^2 \\ \hline & -2x^3 + 8x^2 - 6x \\ & 2x^3 + 2x^2 + 16x \\ \hline & 10x^2 + 10x + 80 \\ & -10x^2 - 10x - 80 \\ \hline & 0x + 0 \end{array}$$

3446. Friction always acts to oppose (potential) motion. The centre of mass of the board would be lower if the LH end of the ladder slipped down. So, friction must be acting up the roof on the LH end.

And, were the LH end to slip downwards, the RH end would slip upwards. Hence, friction must be acting down the roof on the RH end.

— Nota Bene —

For further justification of the above, consider the height of the centre of mass of the board (length b) in the following boundary cases:

① If the board is horizontal, then the centre of mass is $b/2$ above the meeting of the roofs.

② If the board is flat to one of the roofs, then the centre of mass is $b/2 \times \sqrt{2}/2$ above the meeting of the roofs.

Since $b/2 \times \sqrt{2}/2 < b/2$, the board would slip, were the roofs smooth, towards case ②.

3447. The second derivative $h''(x)$ is

$$(2k+1)(2k)x^{2k-1} - (2k+3)(2k+2)x^{2k+1}.$$

This is zero at $x = 0$, as required. We must now show that $h''(x)$ changes sign at $x = 0$. Factorised, the second derivative is

$$\Big((2k+1)(2k) - (2k+3)(2k+2)x^2\Big)x^{2k-1}.$$

At $x = 0$, the first factor has value $(2k+1)(2k)$, which, since $k \in \mathbb{N}$, is non-zero. So, the first factor cannot change sign. The second factor is x^{2k-1}. Since $2k - 1$ is an odd number, this changes sign at $x = 0$. So, $h''(x)$ changes sign at $x = 0$.

Hence, $y = h(x)$ is inflected at $x = 0$.

3448. Using two double-angle formulae, the RHS is

$$\begin{aligned} &\frac{\sin 2x}{\cos 2x + 1} \\ \equiv\ &\frac{2\sin x \cos x}{2\cos^2 x - 1 + 1} \\ \equiv\ &\frac{2\sin x \cos x}{2\cos^2 x} \\ \equiv\ &\frac{\sin x}{\cos x} \\ \equiv\ &\tan x, \text{ as required.} \end{aligned}$$

3449. Solving $X^2 - X < 1$ algebraically gives

$$\tfrac{1}{2}(1-\sqrt{5}) < X < \tfrac{1}{2}(1+\sqrt{5}).$$

Using the normal facility on a calculator,

$$\begin{aligned} &\mathbb{P}\left(\tfrac{1}{2}(1-\sqrt{5}) < X < \tfrac{1}{2}(1+\sqrt{5})\right) \\ =\ &\mathbb{P}\left(X < \tfrac{1}{2}(1+\sqrt{5})\right) - \mathbb{P}\left(X < \tfrac{1}{2}(1-\sqrt{5})\right) \\ =\ &0.94717... - 0.26827... \\ =\ &0.679 \text{ (3sf)}. \end{aligned}$$

3450. Assume, for a contradiction, that the polynomial graph $y = f(x)$ is everywhere convex, and that three points A, B, C on it are collinear. Let their x coordinates be $a < b < c$.

Consider the gradients m_{AB} and m_{BC}. Since the second derivative is positive everywhere, the first derivative (gradient) is increasing everywhere. So, $m_{BC} > m_{AB}$. But A, B, C are collinear, so $m_{BC} = m_{AB}$. This is a contradiction.

Hence, if a polynomial graph $y = f(x)$ is convex for all $x \in \mathbb{R}$, then no three points are collinear. □

3451. Rewriting the integrand,

$$\begin{aligned} &\int \frac{a+x}{b+x}\,dx \\ \equiv\ &\int \frac{a-b+b+x}{b+x}\,dx \\ \equiv\ &\int \frac{a-b}{b+x} + 1\,dx \\ =\ &(a-b)\ln|b+x| + x + c. \end{aligned}$$

— Nota Bene —

This question requires no special treatment of the case $a = b$. In this case, the integrand is 1. And $(a-b)$ is zero, which removes the logarithmic term from the result, giving $x + c$, correctly.

3452. Rearranging and squaring,

$$\cos^2\left(\theta + \tfrac{\pi}{6}\right) = (x - 0.6)^2,$$
$$\sin^2\left(\theta + \tfrac{\pi}{6}\right) = (y - 0.8)^2.$$

Adding these, the first Pythagorean trig identity gives $1 = (x - 0.6)^2 + (y - 0.8)^2$. This is a unit circle centred on $(0.6, 0.8)$. Since $0.6^2 + 0.8^2 = 1$, the circle passes through the origin.

3453. (a) The mouse's position vector, before reaching the first corner, is $0.5\mathbf{i} + t\mathbf{j}$. This gives

$$\tan\theta = \tfrac{t}{0.5} \equiv 2t.$$

(b) Differentiating $\tan\theta = 2t$ with respect to time,

$$\sec^2\theta \tfrac{d\theta}{dt} = 2$$
$$\implies \tfrac{d\theta}{dt} = 2\cos^2\theta.$$

For $\theta \in [0, \pi/2]$, $2\cos^2\theta$ has range $[1, 2]$. So, in radians per second, the lower bound is 1 and the upper bound is 2.

— Nota Bene —

Locations for these bounds are, respectively, the corners (lowest rate of change of angle) and the midpoints of the sides (highest rate of change of angle).

3454. To generate $6x^2$, we need an input transformation $x \mapsto x + k$, which gives

$$y = (x + k)^3 + (x + k).$$

Equating coefficients of x^2, we need $k = 2$:

$$\begin{aligned} y &= (x + 2)^3 + (x + 2) \\ &\equiv x^3 + 6x^2 + 13x + 10. \end{aligned}$$

So, y exceeds the required value by 10. Hence, we also need to apply an output transformation. The overall translation vector is $-2\mathbf{i} - 10\mathbf{j}$.

3455. We factorise and cancel a common factor before taking the limit:

$$\begin{aligned} &\lim_{x\to\ln 2} \frac{e^{2x} - 4}{e^x - 2} \\ &= \lim_{x\to\ln 2} \frac{(e^x + 2)(e^x - 2)}{e^x - 2} \\ &= \lim_{x\to\ln 2} e^x + 2 \\ &= 4. \end{aligned}$$

3456. For SPs, we set the derivative to zero:

$$\begin{aligned} &\cos x - k^3\sec^2 x = 0 \\ \implies &\cos^3 x = k^3 \\ \implies &\cos x = k. \end{aligned}$$

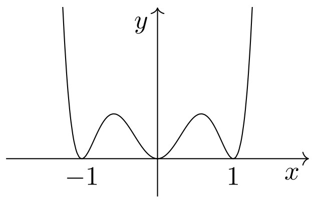
Over $(-\pi/2, \pi/2)$, the range of the cosine function is $(0, 1]$. Hence, if $k \notin (0, 1]$, then the equation $\cos x = k$ has no roots. Hence, $y = \mathrm{f}(x)$ has no SPs. Since f also has no discontinuities over this domain, f is one-to-one, and therefore invertible.

3457. We set up and equation and solve:

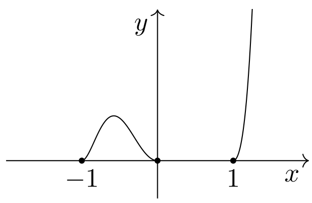
$$\begin{aligned} &\frac{e^c - e^{-c}}{-2 - e^c} = -\tfrac{3}{8} \\ \implies &8e^c - 8e^{-c} = 6 + 3e^c \\ \implies &5e^{2c} - 6e^c - 8 = 0 \\ \implies &e^c = 2, -\tfrac{4}{5}. \end{aligned}$$

We reject the negative value, as e^c is positive. This leaves $c = \ln 2$.

3458. Factorising, $\sqrt{y} = x(x+1)(x-1)$. We now square both sides, noting that, in doing so, we introduce new solutions. This gives $y = x^2(x+1)^2(x-1)^2$. This is a positive sextic curve with double roots at $x = 0, \pm 1$:



But $x^3 - x$ is negative on $(\infty, -1) \cup (0, 1)$. Hence, the graph of $\sqrt{y} = x^3 - x$ contains no points over this domain. This gives



3459. Adding the black counter changes the quantities, but not the $B(2, 1/2)$ binomial probabilities. So,

$$\begin{aligned} \mathbb{P}(\text{BBB in the bag}) &= \tfrac{1}{4} \\ \mathbb{P}(\text{BBW in the bag}) &= \tfrac{1}{2} \\ \mathbb{P}(\text{BWW in the bag}) &= \tfrac{1}{4}. \end{aligned}$$

So, the probabilities of drawing a black counter from the bag are

$$\begin{aligned} \mathbb{P}(\text{BBB, draw B}) &= \tfrac{1}{4} \times 1 = \tfrac{1}{4}, \\ \mathbb{P}(\text{BBW, draw B}) &= \tfrac{1}{2} \times \tfrac{2}{3} = \tfrac{1}{3}, \\ \mathbb{P}(\text{BWW, draw B}) &= \tfrac{1}{4} \times \tfrac{1}{3} = \tfrac{1}{12}. \end{aligned}$$

Hence, given a black counter is drawn out, the probability that two black counters remain is

$$p = \frac{\frac{1}{4}}{\frac{1}{4} + \frac{1}{3} + \frac{1}{12}} = \tfrac{3}{8}.$$

3460. Setting $a = b$ gives $a^n - b^n = 0$. So, according to the factor theorem, $(a - b)$ is a factor of $a^n - b^n$:

$$a^n - b^n \equiv (a - b)(a^{n-1} + a^{n-1}b + ... + b^n).$$

But $a^n - b^n$ is prime, so one of these factors must be equal to 1. It cannot be the latter, so it must be the former: $a - b = 1$ gives $a = b + 1$. □

3461. (a) Yes, the $x^4 = x^2 + 1$ is a quadratic in x^2 with two real roots.

(b) No, the graphs are translations by $\mathbf{j}$.

(c) No, $x^6 + 1 > x^4$ for all $x \in \mathbb{R}$.

3462. The AP has first term $\sin\theta$, and common difference $\cos\theta - \sin\theta$. Hence,

$$\begin{aligned} U_5 &= \sin\theta + 4(\cos\theta - \sin\theta) \\ &\equiv -3\sin\theta + 4\cos\theta. \end{aligned}$$

This can be written in harmonic form $R\cos(\theta+\alpha)$. R is the Pythagorean sum of the coefficients, which is $\sqrt{3^2 + 4^2} = 5$. This is the amplitude of U_5, so it has range $[-5, 5]$, as required.

3463. Let one projectile have initial speed u at angle θ above the horizontal, and the other speed u in the opposite direction, at θ below the horizontal. This gives the horizontal separation as $d_x = 2ut\cos\theta$. The vertical displacements are

$$\begin{aligned} y_1 &= (u\sin\theta)t - \tfrac{1}{2}gt^2 \\ y_2 &= (-u\sin\theta)t - \tfrac{1}{2}gt^2. \end{aligned}$$

So, the vertical separation is $d_y = 2ut\sin\theta$. The distance between the pieces is therefore

$$\begin{aligned} d &= \sqrt{d_x^2 + d_y^2} \\ &= 2ut\sqrt{\cos^2\theta + \sin^2\theta} \\ &\equiv 2ut, \text{ as required.} \end{aligned}$$

— Alternative Method —

Since the projectile pieces both have acceleration g ms^{-2} downwards, this does not affect their relative displacement. The relative speed of projection is $2u$ ms^{-1}. So, after t seconds, they are a distance $d = 2ut$ metres apart, as required.

3464. Using the standard equation $y - y_1 = m(x - x_1)$, all (non-vertical) lines through (a, b) have equation $y - b = m(x - a)$. The gradient m is the derivative $\frac{dy}{dx}$, which gives the DE as

$$\frac{dy}{dx} = \frac{y - b}{x - a}.$$

3465. Consider the prime factors. Since $n \geq 1$, every term A_n has a prime factor of 3. But no term B_n has a prime factor of 3. Hence, there is no number that appears in both sequences.

3466. The statement is false. A counterexample is

$$\begin{aligned} x_1 + x_2 &= 3, \\ 2x_1 + 2x_2 &= 6. \end{aligned}$$

These are the same straight line, and all points on the line satisfy both equations simultaneously.

3467. The small triangle is equilateral. Hence, by similar triangles, the larger triangle formed by the points at the circumference is also equilateral. It has side length 2. Its height is therefore $\sqrt{3}$. The centre divides this height in the ratio $1 : 2$, so the radius of the circle is $\frac{2}{3}\sqrt{3}$. This gives the area as $\frac{4\pi}{3}$.

3468. Solving for intersections,

$$\begin{aligned} & x^4 + (x + 1)^2 = 1 \\ \Longrightarrow\ & 2x^4 + 4x^3 + 6x^2 + 4x = 0 \\ \Longrightarrow\ & x^4 + 2x^3 + 3x^2 + 2x = 0 \\ \Longrightarrow\ & x(x + 1)(x^2 + x + 2) = 0. \end{aligned}$$

The quadratic factor has $\Delta = -7 < 0$, so has no roots. Hence, the curves intersect exactly twice, at $(0, 1)$ and $(-1, 0)$.

3469. Expressing the probabilities in factorials,

$$\begin{aligned} & \mathbb{P}(X = 2) = \mathbb{P}(X = 3) \\ \Longrightarrow\ & \tfrac{n!}{2!(n-2)!} \cdot \tfrac{1}{4}^2 \cdot \tfrac{3}{4}^{n-2} = \tfrac{n!}{3!(n-3)!} \cdot \tfrac{1}{4}^3 \cdot \tfrac{3}{4}^{n-3} \\ \Longrightarrow\ & \tfrac{1}{n-2} \cdot 3^{n-2} = \tfrac{1}{3} \cdot 3^{n-3} \\ \Longrightarrow\ & 3 \cdot 3 = n - 2 \\ \Longrightarrow\ & n = 11. \end{aligned}$$

3470. Solving simultaneously, we substitute for $1 - y^2$, which gives $x^2y = x^2$, thus $x^2(y - 1) = 0$. This has a double root at $x = 0$, signifying that the curves are tangent at the y axis.

— Alternative Method —

The circle has tangents $y = \pm 1$ at the y axis. Both curves pass through $(0, \pm 1)$. Differentiating the other curve implicitly with respect to x,

$$\begin{aligned} & x^2y + y^2 = 1 \\ \Longrightarrow\ & 2xy + x^2\frac{dy}{dx} + 2y\frac{dy}{dx} = 0 \\ \Longrightarrow\ & \frac{dy}{dx} = \frac{-2xy}{x^2 + 2y}. \end{aligned}$$

At $(0, \pm 1)$, the gradient has value 0, so the curves are tangent at the y axis.

3471. (a) The x coordinates are the same when

$$2t - 7 = t \implies t = 7.$$

The y coordinates are 2 and -6 at this time, so the particles do not collide.

(b) The y coordinate of the first particle is 2. This is matched by the second particle at $t = -1$. At this stage, $\mathbf{r}_2 = -\mathbf{i} + 2\mathbf{j} + 7\mathbf{k}$. The first particle has x coordinate -1 at

$$2t - 7 = -1 \implies t = 3.$$

At this time, its z coordinate is $3 \cdot 3 - 2 = 7$. Hence, the paths of the particles intersect.

3472. This is a cubic in e^x:

$$\begin{aligned} &e^{2x} + 12e^{3x} = e^x \\ \implies &12e^{3x} + e^{2x} - e^x = 0 \\ \implies &e^x(12e^{2x} + e^x - 1) = 0 \\ \implies &e^x(4e^x - 1)(3e^x + 1) = 0 \\ \implies &e^x = 0, \tfrac{1}{4}, -\tfrac{1}{3}. \end{aligned}$$

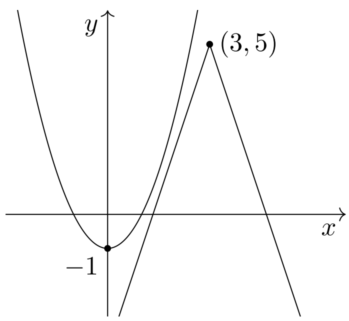
Since $e^x > 0$, we reject the first and last roots, leaving $x = \ln \frac{1}{4}$.

3473. The area beneath the cosine wave is

$$\int_0^{\frac{\pi}{2}} \cos x \, dx = 1.$$

From this, we subtract the area of a quarter-circle of radius 1. This gives the area shaded as $1 - \frac{\pi}{4}$.

3474. Calculating the vertices, the graphs $y = $ LHS and $y = $ RHS have behaviour as follows. We don't yet assume that, as shown, they have no intersections.



To *prove* that the parabola is above the mod graph everywhere, we looks for intersections with the left-hand (active) branch:

$$x^2 - 1 = 5 + 3(x - 3).$$

This is a quadratic with $\Delta = -3$. So, the parabola is always above the mod graph, and thus no values of x satisfy the inequality.

— Nota Bene —

The fact that I have *plotted* these graphs using Tikz in LaTeX does, to some extent, prove the result in question without recourse to algebraic methods. But this approach doesn't translate into handwritten mathematics. The relevant facts are:

- the parabola is positive, with vertex $(0, -1)$,
- the mod graph is negative, with vertex $(3, 5)$, and its active branch has positive gradient.

Neither fact requires a plot, as opposed to a sketch.

3475. We require

$$\begin{aligned} &\left(p\sqrt{2} + q\sqrt{5}\right)^2 = 47 + 6\sqrt{10} \\ \implies &2p^2 + 5q^2 + 2pq\sqrt{10} = 47 + 6\sqrt{10}. \end{aligned}$$

Equating coefficients, $2p^2 + 5q^2 = 47$ and $2pq = 6$. Solving these simultaneously, we take the natural-numbered values, giving $\sqrt{2} + 3\sqrt{5}$.

3476. (a) The DE is

$$\frac{dP}{dt} = kP(q + rt).$$

Naming new constants, this is

$$\frac{dP}{dt} = P(a + bt).$$

Separating the variables,

$$\begin{aligned} &\int \frac{1}{P} \, dP = \int a + bt \, dt \\ \implies &\ln |P| = at + \tfrac{1}{2}bt^2 + c \\ \implies &P = P_0 e^{at + \frac{1}{2}bt^2}. \end{aligned}$$

(b) The DE (as opposed to its solution) has the rate of change of prevalence as the subject:

$$\frac{dP}{dt} = P(a + bt).$$

We assume the prevalence P is positive. Hence, for there to be initial growth, the RHS must be positive, so $a > 0$.

For the solution curve not to predict $P \to \infty$, we then need the rate to become negative. This requires $b < 0$.

(c) The prevalence peaks where $\frac{dP}{dt} = 0$:

$$a + bt = 0 \implies t = -\frac{a}{b}, \text{ as required.}$$

3477. Any non-zero constant of integration produces a counterexample:

$$\mathrm{f}(x) = (x - 1)^2 + 1.$$

The derivative $\mathrm{f}'(x) = 2(x - 1)$ has a factor of $(x - 1)$, but $\mathrm{f}'(x)$ doesn't have a factor of $(x - 1)^2$.

3478. By the product rule,

$$\begin{aligned} & y = x^4 e^{-x} \\ \Longrightarrow\ & \frac{dy}{dx} = \big(4x^3 - x^4\big)e^{-x} \\ \Longrightarrow\ & \frac{d^2y}{dx^2} = \big(12x^2 - 8x^3 + x^4\big)e^{-x}. \end{aligned}$$

So, there are SPs where $4x^3 - x^4 = 0$, which gives $(0, 0)$ and $\big(4, 256e^{-4}\big)$. Substituting these into the second derivative gives 0 and $-64e^{-4}$.

At O, the second derivative test is inconclusive, so further investigation is required. Since the range is $[0, \infty)$, we know that the origin must be a local (and global) minimum.

Hence, the only local maximum is $\big(4, 256e^{-4}\big)$.

3479. Let $z = {}^{y}/_{x}$. This gives a quadratic in z:

$$\begin{aligned} & z - \frac{1}{z} = 1 \\ \Longrightarrow\ & z^2 - z - 1 = 0 \\ \Longrightarrow\ & z = \tfrac{1}{2}\big(1 \pm \sqrt{5}\big). \end{aligned}$$

So, the locus consists of the lines

$$y = \tfrac{1}{2}\big(1 \pm \sqrt{5}\big)x.$$

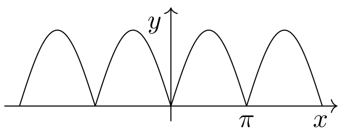
Multiplying the gradients,

$$\tfrac{1}{2}\big(1 + \sqrt{5}\big) \times \tfrac{1}{2}\big(1 - \sqrt{5}\big) = -1.$$

Hence, the lines are perpendicular, as required.

3480. Equating the differences, $b_3 - b_1 = b_4 - b - 1 - b_3$. This simplifies to $2b_3 = b_4$. And, since sequence b_n is geometric, this ratio must appear throughout. Hence, $b_{n+1} = 2b_n$, as required.

3481. The graph $y = |\sin x|$ is periodic, with period π:



The graph has period π. So, the integral of $|\sin x|$ between $x = 0$ and $x = n\pi$ consists of n copies of the integral between 0 and π. This is

$$\begin{aligned} & n\int_0^\pi \sin x \, dx \\ \equiv\ & n\Big[-\cos x\Big]_0^\pi \\ \equiv\ & n\big((1) - (-1)\big) \\ \equiv\ & 2n, \text{ as required.} \end{aligned}$$

3482. Differentiating, $\frac{dy}{dx} = 2ax + b$. So, the tangents are

① $y = (2aq + b)x + k_1$ via $(q, aq^2 + bq + c)$,

② $y = (-2aq + b)x + k_2$ via $(-q, aq^2 - bq + c)$.

Substituting to find the constants k_1 and k_2,

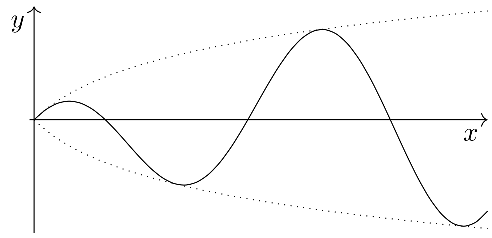
$$\begin{aligned} aq^2 + bq + c &= (2aq + b)q + k_1, \\ aq^2 - bq + c &= (-2aq + b) \cdot -q + k_2. \end{aligned}$$

Each of these simplifies to the same value

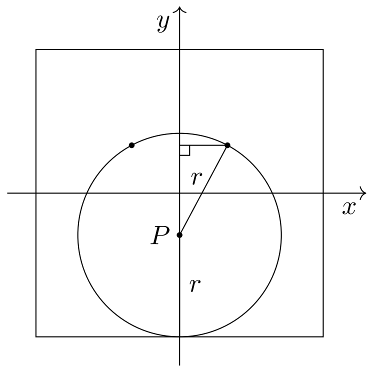
$$k_1 = k_2 = -aq^2 + c.$$

Therefore, since the tangent lines have the same y intercept, they must cross on the y axis.

3483. The graph $y = \ln(x+1)$, when multiplied by $\cos x$, oscillates between the graphs $y = \ln(x + 1)$ and $y = -\ln(x + 1)$. Sketching these as dotted lines, the curve $y = \ln(x + 1)\cos x$ is



3484. By symmetry, the largest circle must have a centre at P as in the diagram. Call the radius r.



The points $(\pm{}^1/_3, {}^1/_3)$ are a y distance $\frac{4}{3}$ from the bottom edge. So, we require

$$\begin{aligned} & r + \sqrt{r^2 - \tfrac{1}{3}^2} = \tfrac{4}{3} \\ \Longrightarrow\ & \sqrt{r^2 - \tfrac{1}{9}} = \tfrac{4}{3} - r \\ \Longrightarrow\ & r^2 - \tfrac{1}{9} = \tfrac{16}{9} - \tfrac{8}{3}r + r^2 \\ \Longrightarrow\ & r = \tfrac{17}{24}. \end{aligned}$$

3485. By the quotient rule,

$$\begin{aligned} & \frac{d}{dx}\left(\frac{\sin x}{2 + \cos x}\right) \\ \equiv\ & \frac{\cos x(2 + \cos x) + \sin^2 x}{(2 + \cos x)^2} \\ \equiv\ & \frac{1 + 2\cos x}{(2 + \cos x)^2}. \end{aligned}$$

Using the quotient rule again gives the following, in which the fraction is split over two lines:

$$\begin{aligned}&\frac{d}{dx}\left(\frac{1+2\cos x}{(2+\cos x)^2}\right)\\&\quad=\frac{-2\sin x(2+\cos x)^2}{(2+\cos x)^4}\\&\qquad-\frac{(1+2\cos x)\cdot 2(2+\cos x)\cdot -\sin x}{(2+\cos x)^4}.\end{aligned}$$

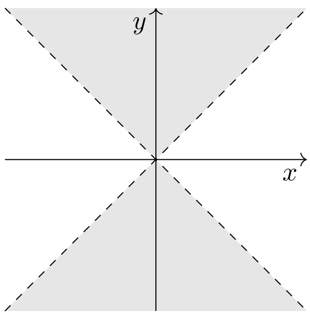
Cancelling a common factor of $(2+\cos x)$ on the top and bottom, this is

$$\begin{aligned}&\frac{-2\sin x(2+\cos x)-(1+2\cos x)\cdot -2\sin x}{(2+\cos x)^3}\\&\equiv\frac{2\sin x(-2-\cos x+1+2\cos x)}{(2+\cos x)^3}\\&\equiv\frac{2\sin x(\cos x-1)}{(2+\cos x)^3},\text{ as required.}\end{aligned}$$

3486. The boundary equation gives a pair of lines:

$$\begin{aligned}&(x+y)(x-y)=0\\\implies &y=\pm x.\end{aligned}$$

For the product $(x+y)(x-y)$ to be negative, then, precisely one of the factors $(x+y)$ and $(x-y)$ must be be negative:



3487. Starting with the first Pythagorean identity,

$$\begin{aligned}&\sin^2\theta+\cos^2\theta\equiv 1\\\implies &\operatorname{cosec}^2\theta\equiv\frac{1}{1-\cos^2\theta}\\\implies &\operatorname{cosec}\theta=\frac{\pm 1}{\sqrt{1-\cos^2\theta}}\\&\qquad\equiv\frac{\pm\sec\theta}{\sqrt{\sec^2\theta-1}}.\end{aligned}$$

3488. The probability that the last letter is A is $\frac{2}{8}$. Then the probability that the second last letter is also A is $\frac{1}{7}$. The rest of the letters don't matter. This gives the probability that the rearrangement ends AA as

$$p=\tfrac{2}{8}\times\tfrac{1}{7}=\tfrac{1}{28}.$$

— Alternative Method —

For a combinatorial approach, take the 8 letters as distinct. The possibility space consists of 8! orders. Of these, $2!\times 6!$ finish in AA. So, the probability in question is

$$p=\tfrac{2!\times 6!}{8!}=\tfrac{1}{28}.$$

3489. First, we solve $\tan x=\sqrt{3}$, for $x\in[0,\pi]$. There is one root: $x=\pi/3$. Introducing the mod function gives another root at $x=-\pi/3$. So, the solution is $x\in\{-\pi/3,\pi/3\}$.

3490. When multiplied out, $(ax+b)(bx+c)(cx+d)$ has leading coefficient $abc=6$. So, at least one of a,b,c is equal to 1. Hence, there is an integer root. This cannot be $x=-1$, since a,b,c,d are distinct. The next option is $x=-2$. Testing this, it is a root, so $(x+2)$ is a factor. Taking this out, the factorisation is

$$\mathrm{f}(x)=(x+2)(2x+3)(3x+4).$$

So, the roots are $x=-2,-3/2,-4/3$.

3491. (a) $\mathrm{f}(x)=x^2\implies\mathrm{f}''(x)=2$.

(b) By the chain rule,

$$\mathrm{g}'(x)=\frac{x}{\sqrt{R^2-x^2}}.$$

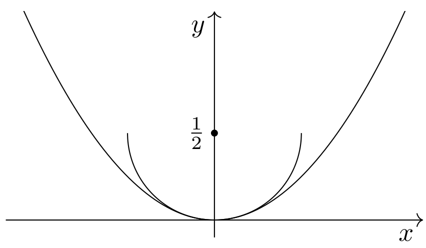
Then, by the quotient rule,

$$\begin{aligned}\mathrm{g}''(x)&=\frac{\sqrt{R^2-x^2}+x^2\left(R^2-x^2\right)^{-\frac{1}{2}}}{R^2-x^2}\\&\equiv\frac{R^2}{\left(R^2-x^2\right)^{\frac{3}{2}}}.\end{aligned}$$

So, $\mathrm{g}''(0)=\frac{1}{R}$.

(c) We need $\frac{1}{R}=2$, giving $R=\frac{1}{2}$.

(d) The curve $y=x^2$ is best approximated at the origin by a circle centred at $(0,1/2)$, with radius $1/2$. This is

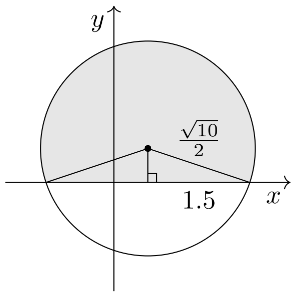


3492. A cubic y has a derivative $\frac{dy}{dx}$ which is quadratic. Squaring this, the LHS of the DE is a quartic plus a cubic, so has degree 4. The RHS is linear, with degree 1. This is not possible. Hence, no cubic satisfies the DE.

3493. The circle has centre $(1/2, 1/2)$ and radius $\sqrt{10}/2$. Solving for x intercepts,

$$(2x-1)^2+1=10$$
$$\Longrightarrow x=-1,2.$$

The distance between these is 3. Splitting this in two gives right-angled triangles:



So, the angle subtended at the centre of the circle by the (unshaded) minor segment is

$$2\arcsin\frac{1.5}{\frac{\sqrt{10}}{2}}=2.4980\text{... radians.}$$

The area of the minor segment is

$$\tfrac{1}{2}\cdot\tfrac{5}{2}(2.4980-\sin 2.4980)=2.3724.$$

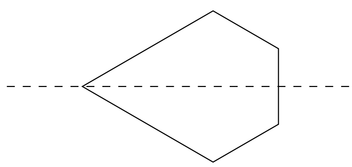
Hence, the area of the shaded major segment is

$$\tfrac{5}{2}\pi-2.3724=5.48\text{ (3sf).}$$

3494. The student has confused *increasing* with *convex*.

- *Increasing* means that $f(x)$ is increasing with x, i.e. that $f'(x)>0$. If $f'(x)>0$, then chords may be above or below $y=f(x)$, so the trapezium rule may over or underestimate.
- *Convex* means that $f'(x)$ is increasing with x, i.e. that $f''(x)>0$. If $f''(x)>0$, then chords will always be above the curve $y=f(x)$, so the trapezium rule will overestimate.

3495. The two sides of length 2 share a vertex, which must be on the line of symmetry. The angle 60° fixes three of the vertices. The remaining two must be symmetrical. Since there are no reflex interior angles, the pentagon must be



The interior angles are 60° and 120°. So, the area is that of two equilateral triangles of side length 2, minus that of one equilateral triangle of side length 1. This gives

$$A_{\text{pentagon}}=2\times\tfrac{4\sqrt{3}}{4}-\tfrac{\sqrt{3}}{4}=\tfrac{7\sqrt{3}}{4}.$$

3496. At $x=0$, both cosines have value 1. Each is individually maximised, so their sum is too, at $h(0)=2$. At $x=\pi$, the arguments of both cosines are $n\pi$, where n is an odd number, so both cosines have value -1. Each is individually minimised, so their sum is too, at $h(\pi)=-2$. This gives the range of h as $[-2,2]$.

— Nota Bene —

This type of question is, in general, very hard. If the question had involved $h(x)=\sin ax+\cos bx$, say, then some heavy calculus would be involved. The range of such functions is $[-2+\varepsilon, 2-\varepsilon]$, where ε is small but non-zero.

3497. The angle between planes is the angle between their normals. The normals in question are a space diagonal of length $\sqrt{3}$ and an edge of length 1. These form a $(1,\sqrt{2},\sqrt{3})$ right-angled triangle. So, the angle is $\arccos\frac{1}{\sqrt{3}}=54.7°$ (1dp).

3498. (a) Horizontally, $x=\sqrt{10}t$. Vertically, $y=1-5t^2$. Substituting for t gives $y=1-\frac{1}{2}x^2$.

(b) Setting $y=0$, we have $1=\frac{1}{2}x^2$, so $x=\pm\sqrt{2}$. Hence, the distance between bounces is $2\sqrt{2}$.

(c) The new parabola is a translation of the first by $2\sqrt{2}\mathbf{i}$. So, it has equation

$$y=1-\tfrac{1}{2}\left(x-2\sqrt{2}\right)^2$$
$$\equiv -3+2\sqrt{2}x-\tfrac{1}{2}x^2,\text{ as required.}$$

3499. (a) Using the quotient rule,

$$\frac{dy}{dx}=\frac{-\sin x(1-\sin x)+\cos^2 x}{(1-\sin x)^2}$$
$$\equiv\frac{1-\sin x}{(1-\sin x)^2}$$
$$\equiv\frac{1}{1-\sin x}.$$

(b) At $(0,1)$, $\frac{dy}{dx}=1$. Hence, the equation of the normal is $y-1=-x$, which is $y=1-x$.

(c) The normal intersects the curve again where

$$\frac{\cos x}{1-\sin x}=1-x$$
$$\Longrightarrow\frac{\cos x}{1-\sin x}+x-1=0.$$

The Newton-Raphson iteration is

$$x_{n+1}=x_n-\frac{\frac{\cos x_n}{1-\sin x_n}+x_n-1}{\frac{1}{1-\sin x_n}+1}.$$

With $x_0=2.5$, we get $x_n\to 2.657$ (4sf).

3500. These are two unit spheres, with centres at $(0, 0, 0)$ and $(3, 4, 0)$. The shortest path between them lies normal to both; this is along the line of centres. By Pythagoras, the distance between the centres is 5. Subtracting two radii from this, the shortest distance between the spheres is 3.

——— End of 35th Hundred ———